

# Backreaction as an alternative to dark energy and modified gravity

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**ABSTRACT:** The predictions of homogeneous and isotropic cosmological models with ordinary matter and gravity are off by a factor of two in the late universe. One possible explanation is the known breakdown of homogeneity and isotropy due to the formation of non-linear structures. We review how inhomogeneities affect the average expansion rate and can lead to acceleration, and consider a semi-realistic model where the observed timescale of ten billion years emerges from structure formation. We also discuss the relation between the average expansion rate and observed quantities.

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## 1. Introduction

### 1.1 A factor of 2

The early universe, at least from Big Bang Nucleosynthesis at one second onwards, is well described by a model which is exactly homogeneous and isotropic (up to linear perturbations) and contains only ordinary matter, and where the relation between matter and spacetime geometry is given by ordinary general relativity. Here ordinary matter means that the pressure is non-negative, and ordinary general relativity is based on the four-dimensional Einstein-Hilbert action. Such a model also works well when applied to the late universe, except that the distance and the expansion rate are underpredicted by a factor of two. The observed distance to the last scattering surface at redshift 1090 is a factor of 1.4–1.7 longer than predicted by the spatially flat homogeneous and isotropic model dominated by pressureless matter, keeping the Hubble parameter today fixed (and assuming a power-law spectrum of primordial

perturbations) [1]. Observations of type Ia supernovae and large-scale structure are consistent with this cosmic microwave background (CMB) measurement, and they show that the discrepancy arises at redshifts of order one and smaller, when the universe is about ten billion years old. In the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models, the explanation for the longer distances is simple: the expansion of the universe has accelerated, so objects have receded further away.

Most cosmological observations probe distances, but there are also some measurements of the expansion rate. Galaxy ages [2] and the radial baryon acoustic oscillation signal [3] have been used to measure the expansion rate as a function of redshift, and the value of the Hubble parameter today is known with some accuracy [4]. The expansion rate observations agree well with the distance observations and support the interpretation of faster expansion being the cause of the longer distances. The measured Hubble parameter is about a factor of 2 larger than expected compared to the matter density,  $\Omega_{m0} \equiv 8\pi G_N \rho_{m0}/(3H_0^2) \approx 0.25$  [5], or a factor of 1.2–1.5 larger if compared to the age of the universe,  $H_0 t_0 \approx 0.8$ –1 instead of  $H_0 t_0 = 2/3$  [6].

As the observations are beyond reasonable doubt, at least one of the three assumptions of the theoretical model is wrong. Either there is exotic matter with negative pressure (dark energy), general relativity does not hold on cosmological scales, or the homogeneous and isotropic approximation is not valid at late times.

Apart from observations of the expansion rate and the distance scale, there is no evidence for negative-pressure matter or modifications of general relativity. For example, such matter has not been detected in the laboratory, nor have deviations from general relativity been observed in the Solar system<sup>1</sup>. Likewise, no objects have been seen to accelerate away from each other. All of the relevant observations involve averages over large volumes or integrals over large distances. The situation is different from that of dark matter, for which there is evidence from several kinds of observations on various scales and eras, including local physics, such as the CMB peak structure, large-scale structure, rotation curves of galaxies, the motions of galaxies and gas in clusters, gravitational lensing and so on [8]. This is the reason why constructing alternatives to dark matter requires resort to baroque models, if it is possible at all [9]. In contrast, the various observations usually interpreted as indicating dark energy or modified gravity are all different tracers of the same quantity: longer distances and faster expansion. An effect which leads to faster expansion and correspondingly increases the distances can explain all of these observations. (Indeed, the only effect of the favorite dark energy candidate, vacuum energy, is to change the expansion rate.)

Unlike the assumptions of ordinary matter and gravity, the assumption of only linear deviations from homogeneity and isotropy is known to be violated at late times

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<sup>1</sup>Apart from possibly the Pioneer anomaly and the flyby anomaly [7].

due to the formation of non-linear structures, a process which has a preferred time of about ten billion years. Before concluding that new physics is needed to explain the observations, we should therefore study the possibility that the factor two failure of the predictions of the simple homogeneous and isotropic models is due to the known breakdown of homogeneity and isotropy [10–15].

## 1.2 Our clumpy universe

It is important to distinguish between *exact* homogeneity and isotropy and *statistical* homogeneity and isotropy. Exact homogeneity and isotropy means that the space has a local symmetry: all points and all directions are equivalent. Statistical homogeneity and isotropy simply means that if we consider a box anywhere in the universe, the mean quantities in the box do not depend on its location, orientation or size, provided that it is larger than the homogeneity scale.

The early universe is nearly exactly homogeneous and isotropic, in two ways. First, the amplitude of the perturbations around homogeneity and isotropy is small. Second, the distribution of the perturbations is statistically homogeneous and isotropic. At late times, when density perturbations become non-linear, the universe is no longer locally near homogeneity and isotropy, and there are deviations of order unity in quantities such as the local expansion rate. However, the distribution of the non-linear regions remains statistically homogeneous and isotropic on large scales. The homogeneity scale appears to be around 100 Mpc today [16] (though see [17]).

Due to the statistical symmetry, the average expansion rate evaluated inside each box is equal (up to statistical fluctuations), but this does not mean that it would be the same as in a completely smooth spacetime, because there are structures in the box. The feature that the average evolution of a clumpy space is not the same as the evolution of a smooth space is called *backreaction* [18–20]. We can say that time evolution and averaging do not commute: if we smooth a clumpy distribution and calculate the time evolution of the smooth quantities with the Einstein equation, the result is not the same as if we evolved the full clumpy distribution and took the average at the end. Put simply, FRW models describe universes which are exactly homogeneous and isotropic. They do not describe universes which are only statistically homogeneous and isotropic. The effect of clumpiness on the average was first discussed in detail by George Ellis in 1983 under the name *fitting problem* [21]. Clumpiness affects the expansion of the universe, the way light propagates in the universe and the relationship between the two. The possibility that the late time deviations from the simple homogeneous and isotropic models would be explained in terms of these changes due to structure formation can be called the *backreaction conjecture*.

In section 2 we go through the basics of how structures affect the expansion rate. An increased average expansion rate due clumpiness may be a bit unfamiliar concept, so in section 3 we explain what average acceleration means physically using

a simple toy model, and in section 4 we go on to discuss a semi-realistic model for the universe where we can get some numbers out. We also briefly mention the relation of the backreaction problem to the fascinating question of the Newtonian limit of general relativity. In section 5 we discuss the relation of the average expansion rate to light propagation. The order is a bit backwards, as it is the observed redshifts and distances which are the important quantities, but it is perhaps easier to start from the expansion rate. We conclude in section 6 with a summary.

## 2. Backreaction, exactly

### 2.1 The local expansion rate

We consider a universe where the energy density of matter dominates over the pressure, anisotropic stress and energy flux everywhere. In other words, the matter can be considered a pressureless ideal fluid, or dust. We also assume that the relation between the matter and the geometry is given by the Einstein equation:

$$G_{\alpha\beta} = 8\pi G_N T_{\alpha\beta} = 8\pi G_N \rho u_\alpha u_\beta , \quad (2.1)$$

where  $G_{\alpha\beta}$  is the Einstein tensor,  $G_N$  is Newton's constant,  $T_{\alpha\beta}$  is the energy-momentum tensor,  $\rho$  is the energy density and  $u^\alpha$  is the velocity of observers co-moving with the dust.

The evolution and constraint equations can be written elegantly in terms of the gradient of  $u_\alpha$  and the electric and magnetic components of the Weyl tensor [22],

$$\nabla_\beta u_\alpha = \frac{1}{3} h_{\alpha\beta} \theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} , \quad (2.2)$$

where  $h_{\alpha\beta}$  projects orthogonally to  $u^\alpha$ : roughly speaking, if  $u^\alpha$  is understood as the time direction, then  $h_{\alpha\beta}$  spans the spatial directions<sup>2</sup>. The trace  $\theta \equiv \nabla_\alpha u^\alpha$  is the volume expansion rate, the traceless symmetric part  $\sigma_{\alpha\beta}$  is the shear tensor and the antisymmetric part  $\omega_{\alpha\beta}$  is the vorticity tensor. For an infinitesimal fluid element,  $\theta$  indicates how its volume changes in time, keeping the shape and the orientation fixed, while the shear changes the shape and vorticity changes the orientation. Roughly speaking,  $\theta$  is the time derivative of the volume element divided by the volume element. In the FRW case, the volume expansion rate is just  $3H$ , where  $H$  is the Hubble parameter.

The equations can be decomposed into scalar, vector and tensor parts with respect to the spatial directions orthogonal to  $u^\alpha$ . We need only the scalar parts (we omit a scalar equation related to the vorticity, which we don't need),

$$\dot{\theta} + \frac{1}{3}\theta^2 = -4\pi G_N \rho - 2\sigma^2 + 2\omega^2 \quad (2.3)$$

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<sup>2</sup>Strictly speaking, if the vorticity does not vanish, there are no hypersurfaces orthogonal to  $u^\alpha$ .

$$\frac{1}{3}\theta^2 = 8\pi G_N \rho - \frac{1}{2} {}^{(3)}R + \sigma^2 - \omega^2 \quad (2.4)$$

$$\dot{\rho} + \theta\rho = 0, \quad (2.5)$$

where a dot stands for derivative with respect to proper time  $t$  measured by observers comoving with the dust,  $\sigma^2 \equiv \frac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta} \geq 0$  and  $\omega^2 \equiv \frac{1}{2}\omega^{\alpha\beta}\omega_{\alpha\beta} \geq 0$  are the shear and the vorticity scalar, respectively, and in the irrotational case,  ${}^{(3)}R$  is the spatial curvature of the hypersurface orthogonal to  $u^\alpha$ ; see [23] for the definition in the case of non-vanishing vorticity.

The equation (2.5) shows that the energy density is proportional to the inverse of the volume, in other words that mass is conserved. The second equation (2.4) is the local equivalent of the Friedmann equation, and it relates the expansion rate to the energy density, spatial curvature, shear and vorticity. The left-hand side of (2.3) gives the local acceleration. Let us assume that the fluid is irrotational, i.e. that the vorticity is zero. Taking into account vorticity would lead to technical complications which we want to avoid [24]. As vorticity contributes positively to the acceleration, putting it to zero gives a lower bound on the acceleration. With no vorticity, the local acceleration is always negative, or at most zero. This is just an expression of the fact that gravity is attractive for matter satisfying the strong energy condition. One might think that if we are interested in accelerated expansion<sup>3</sup>, there is no point in proceeding further. If the local expansion rate decelerates everywhere and at all times, surely the average expansion rate will also decelerate? However, this conclusion is false.

## 2.2 The average expansion rate

Let us consider the average expansion rate. The spatial average of a scalar quantity  $f$  is simply the integral over the spatial hypersurface orthogonal to  $u^\alpha$  (this is also the hypersurface of constant proper time  $t$  measured by the observers), with the correct volume element, divided by the volume:

$$\langle f \rangle(t) \equiv \frac{\int d^3x \sqrt{{}^{(3)}g(t, \bar{x})} f(t, \bar{x})}{\int d^3x \sqrt{{}^{(3)}g(t, \bar{x})}}, \quad (2.6)$$

where  ${}^{(3)}g$  is the determinant of the metric on the hypersurface of constant proper time  $t$ .

Averaging (2.3)–(2.5), we obtain the Buchert equations [25]

$$3\frac{\ddot{a}}{a} = -4\pi G_N \langle \rho \rangle + \mathcal{Q} \quad (2.7)$$

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<sup>3</sup>Note that inferring from the observed distances that the expansion has accelerated depends on assuming the FRW relation between distance and the expansion rate. If backreaction is important, this relation is modified; see section 5. Based on direct measurements of the expansion rate, we can only say that there has been less deceleration, not that the expansion has accelerated.

$$3\frac{\dot{a}^2}{a^2} = 8\pi G_N \langle \rho \rangle - \frac{1}{2} \langle {}^{(3)}R \rangle - \frac{1}{2} \mathcal{Q} \quad (2.8)$$

$$\partial_t \langle \rho \rangle + 3\frac{\dot{a}}{a} \langle \rho \rangle = 0, \quad (2.9)$$

where the backreaction variable  $\mathcal{Q}$  contains the effect of inhomogeneity and anisotropy,

$$\mathcal{Q} \equiv \frac{2}{3} (\langle \theta^2 \rangle - \langle \theta \rangle^2) - 2\langle \sigma^2 \rangle, \quad (2.10)$$

and the scale factor  $a(t)$  is defined so that the volume of the spatial hypersurface is proportional to  $a(t)^3$ ,

$$a(t) \equiv \left( \frac{\int d^3x \sqrt{{}^{(3)}g(t, \bar{x})}}{\int d^3x \sqrt{{}^{(3)}g(t_0, \bar{x})}} \right)^{\frac{1}{3}}, \quad (2.11)$$

where  $a$  has been normalised to unity at time  $t_0$ , which we take to be today. As  $\theta$  gives the expansion rate of the volume, this definition of  $a$  is equivalent to  $3\dot{a}/a \equiv \langle \theta \rangle$ . We also use the notation  $H \equiv \dot{a}/a$ .

The equations (2.7)–(2.9) were first derived by Thomas Buchert in 1999 [25]. They have a slightly different physical interpretation than the FRW equations due to the different meaning of the scale factor. In FRW models, the scale factor is a component of the metric, and indicates how the space is evolving locally. In the present context,  $a(t)$  does not describe local behaviour, and it is not part of the metric. It simply gives the total volume of a region.

Mathematically, the Buchert equations differ from the FRW equations by the presence of the backreaction variable  $\mathcal{Q}$  and the related feature that the average spatial curvature can have non-trivial evolution. In the FRW case,  $\mathcal{Q} = 0$ , and it then follows from the integrability condition between (2.7) and (2.8) that  $\langle {}^{(3)}R \rangle \propto a^{-2}$ . In general,  $\mathcal{Q}$  is non-zero, and it expresses the non-commutativity of time evolution and averaging. The backreaction variable  $\mathcal{Q}$  has two parts: the second term in (2.10) is the average of the shear scalar, which is also present in the local equations (2.3)–(2.5). It is always negative (unless the spacetime is FRW, in which case it is zero), and acts to decelerate the expansion. In contrast, the first term in (2.10), the variance of the expansion rate, has no local counterpart. It may be called emergent in the sense that it is purely a property of the average system. The variance is always positive (unless the expansion is homogeneous, in which case it is zero). If the variance is sufficiently large compared to the shear and the energy density, the average expansion rate accelerates according to (2.7), even though (2.3) shows that the local expansion rate decelerates everywhere.

### 3. Understanding acceleration

#### 3.1 A two-region toy model

It may seem paradoxical that the average expansion rate accelerates even though the

local expansion rate slows down everywhere, so we first look at a simple toy model to understand the physical meaning, before venturing to a more realistic model. Let us give the punchline right away. In an inhomogeneous space, different regions expand at different rates. Regions with faster expansion rate increase their volume more rapidly, by definition. Therefore the fraction of volume in faster expanding regions rises, so the average expansion rate can rise. Whether the average expansion rate actually does rise depends on how rapidly the fraction of fast regions grows relative to the rate at which their expansion rate is decreasing.

In the real universe, the initial distribution of density (and thus of the expansion rate) is very smooth, with only small local variations. In a simplified picture, overdense regions slow down more as their density contrast grows, and eventually they turn around and collapse to form stable structures. Underdense regions become ever emptier, and their deceleration decreases. Regions thus become more differentiated, and the variance of the expansion rate grows.

We can illustrate this with a simple toy model where there are two spherically symmetric regions, one underdense and one overdense [19, 26]. We consider the regions to be Newtonian, so their evolution is given by the spherical collapse model and the underdense equivalent, i.e. they expand like dust FRW universes with negative and positive spatial curvature, respectively.

We denote the scale factors of the underdense and the overdense region by  $a_1$  and  $a_2$ , respectively. We take the underdense region, which models a cosmological void, to be completely empty, so it expands like  $a_1 \propto t$ . The evolution of the overdense region, which models the formation of a structure such as a cluster, is given by  $a_2 \propto 1 - \cos \phi$ ,  $t \propto \phi - \sin \phi$ , where the parameter  $\phi$  is called the development angle. The value  $\phi = 0$  corresponds to the big bang singularity, from which the overdense region expands until  $\phi = \pi$ , when it turns around and starts collapsing. The region shrinks to zero size at  $\phi = 2\pi$ . In studies of structure formation, the collapse is usually taken to stabilise at  $\phi = 3\pi/2$  due to vorticity and velocity dispersion, and we also follow the evolution only up to that point. The total volume is  $a^3 = a_1^3 + a_2^3$ . The average expansion rate and acceleration are

$$H = \frac{a_1^3}{a_1^3 + a_2^3} H_1 + \frac{a_2^3}{a_1^3 + a_2^3} H_2 \equiv v_1 H_1 + v_2 H_2 \quad (3.1)$$

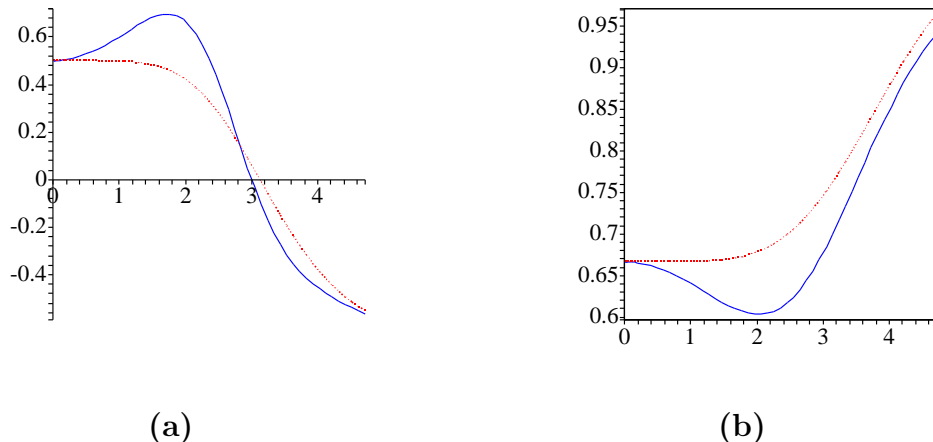
$$\frac{\ddot{a}}{a} = v_1 \frac{\ddot{a}_1}{a_1} + v_2 \frac{\ddot{a}_2}{a_2} + 2v_1 v_2 (H_1 - H_2)^2. \quad (3.2)$$

The average expansion rate is the volume-weighted average of the expansion rates  $H_1$  and  $H_2$ , as one would expect. It is therefore bounded from above by the fastest local expansion rate. However, from the fact that both  $H_1$  and  $H_2$  decrease it does not follow that their weighted average would decrease, or that the average expansion rate would decelerate<sup>4</sup>. This is illustrated by the acceleration equation

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<sup>4</sup>Of course, an increase in  $H$  is not required for acceleration.





**Figure 1:** The evolution of the toy model as a function of the development angle  $\phi$ . (a): The deceleration parameter  $q$  in the toy model (blue, solid) and in the  $\Lambda$ CDM model (red, dash-dot). (b): The Hubble parameter multiplied by time,  $Ht$ , in the toy model (blue, solid) and in the  $\Lambda$ CDM model (red, dash-dot).

(3.2). The first two terms are the volume-weighted average, and because the regions decelerate (or at most have zero acceleration, in the completely empty case), it is negative. However, there is an additional term related to the difference between the two expansion rates, which is always positive (as long as the regions have non-zero volume and different expansion rates). This term arises because a time derivative of (3.1) operates not only on  $H_1$  and  $H_2$ , but also on  $v_1$  and  $v_2$ . In terms of the general acceleration equation (2.7), the first two terms in (3.2) come from the average density, and the last term is the backreaction variable  $\mathcal{Q}$ .

The toy model has one free parameter, the relative size of the two regions at some time. For illustration purposes, we fix this by setting the deceleration parameter  $q \equiv -\ddot{a}/a/H^2$  at  $\phi = 3\pi/2$  to the value of the spatially flat  $\Lambda$ CDM FRW model with  $\Omega_\Lambda = 0.7$ . In figure 1 (a) we plot  $q$  as a function of the development angle  $\phi$ . In addition to the toy model, we show the  $\Lambda$ CDM model for comparison.

The  $\Lambda$ CDM model starts matter-dominated, with  $q = 1/2$ . As vacuum energy becomes important, the model decelerates less and then crosses over to acceleration. Asymptotically,  $q$  approaches  $-1$  from above as the Hubble parameter approaches a constant. The backreaction model also starts with the FRW matter-dominated behaviour, then the expansion slows down more, before  $q$  turns around and the expansion decelerates less and eventually accelerates: in fact the acceleration is stronger than in the  $\Lambda$ CDM model.

The acceleration is not due to regions speeding up locally, but due to the slower region becoming less represented in the average. First the overdense region brings down the expansion rate, but its fraction of the volume falls because of the slower expansion, so eventually the underdense region takes over and the average expansion

rate rises. This is particularly easy to understand after the overdense region has started collapsing at  $\phi = \pi$ . Then the contribution  $v_2 H_2$  of the overdense region to (3.1) is negative, and its magnitude shrinks rapidly as  $v_2$  decreases, so it is transparent that the expansion rate increases. Note that while there is an upper bound on the expansion rate, there is no lower bound on the collapse rate. Therefore, the acceleration can be arbitrarily rapid, and  $q$  can even reach minus infinity in a finite time. (This simply means that the collapsing region becomes so dominant that  $H^2$  vanishes in the denominator of  $q$ .) This is in contrast to FRW models, where  $q \geq -1$  unless the null energy condition (or the modified gravity equivalent) is violated. After the overdense region stops being important, the expansion rate will be given by the underdense region alone, and the expansion will again decelerate. Acceleration is a transient phenomenon associated with the volume becoming dominated by the underdense region.

Figure 1 (b) shows the Hubble parameter multiplied by time as a function of the development angle  $\phi$ . This contains the same information as figure 1 (a), but plotted in terms of the first derivative of the scale factor instead of the second derivative. In the  $\Lambda$ CDM model,  $Ht$  starts from  $2/3$  in the matter-dominated era and increases monotonically without bound as  $H$  approaches a constant. In the toy model,  $Ht$  falls as the overdense region slows down, then rises as the underdense region takes over, approaching unity from below. The Hubble parameter in the toy model is smaller than in the  $\Lambda$ CDM model at all times, and because  $H$  is bounded from above by the fastest local expansion rate,  $Ht$  cannot exceed unity. This bound also holds in realistic models: as long as the matter can be treated as dust and vorticity can be neglected, we have  $Ht \leq 1$  at all times [27], in contrast to FRW models with exotic matter or modified gravity. This is a prediction of backreaction. (For discussion of vorticity and non-dust terms in the energy-momentum tensor, see [24, 28].)

Whether the expansion accelerates depends on how rapidly the faster expanding regions catch up with the slower ones, roughly speaking by how steeply the  $Ht$  curve rises. This is why the variance contributes positively to acceleration: the larger the variance, the bigger the difference between fast and slow regions, and the more rapidly the fast regions take over.

## 4. Towards reality

### 4.1 A statistical semi-realistic model

The toy model shows how acceleration due to inhomogeneities can occur and makes transparent what this means physically. Acceleration has also been demonstrated with the exact spherically symmetric dust solution, the Lemaître-Tolman-Bondi model [29–31]. So there is no ambiguity: accelerated average expansion due to inhomogeneities is possible. The question is whether the distribution of structures in

the universe is such that this mechanism is realised. The statement that faster expanding regions increase their volume more rapidly makes it sound as if there would necessarily be less deceleration (if not acceleration) than in the FRW case. For a set of isolated regions, this is true: eventually, the volume will be dominated by the fastest region. However, the characteristic feature of structures in the real universe is their hierarchical buildup. Smaller structures become incorporated into larger ones, and rapidly expanding voids can be extinguished by collapsing clouds.

The non-linear evolution of structures is too complex to follow exactly. However, because the universe is statistically homogeneous and isotropic, statistical properties are enough to evaluate the average expansion rate. In terms of the Buchert equations (2.7)–(2.9), the average expansion rate is completely determined by the variance and the average shear scalar. If one wants an upper limit on the acceleration, discarding the shear and looking only at the variance would be enough. The average expansion rate is also determined if we know which fraction of the universe is in which state of expansion or collapse. Instead of trying to find a solution for the metric and calculating the quantities of interest from it, it is useful to consider an ensemble of regions from which we can determine the average expansion rate without having to consider the global metric. We now discuss a semi-realistic model which does this by extending the two fixed regions of the toy model to a continuous distribution of regions which evolves in time [32, 33].

The starting point is the spatially flat matter-dominated FRW model with a linear Gaussian field of density fluctuations. Structure formation, even though complicated, is a deterministic process. Therefore any statistical quantity at late times is determined by the initial distribution processed by gravity. For a Gaussian distribution, the power spectrum contains all information. So even in the completely non-linear regime, the average expansion rate follows from the power spectrum. The problem is formulating a tractable model for propagating the structures given by the initial power spectrum into the non-linear regime with gravity. One approach, proposed in [34], is to identify structures at late times with spherical peaks in the original linear density field, smoothed on the appropriate scale. The number density of peaks as a function of the smoothing scale and peak height can be determined analytically in terms of the power spectrum. In the original application, the correspondence between peaks and structures was assumed to hold only for very non-linear overdense structures: all peaks exceeding a certain density threshold were identified with stabilised structures. Here the idea is a bit different: spherical peaks of any density are identified with structures having the same linear density contrast. Troughs are identified with spherical voids in the same way. (As the distribution is Gaussian, the statistics of peaks and troughs are identical.) We keep the smoothing threshold fixed such that  $\sigma(t, R) = 1$ , where  $\sigma$  is the root mean square linear density contrast,  $t$  is time and  $R$  is the smoothing scale. Non-linear structures form at  $\sigma \approx 1$ , so  $R$  corresponds to the size of the typical largest structures, and grows in time. The

smoothing is just a simplified treatment of the complex stabilisation and evolution of structures in the process of hierarchical structure formation.

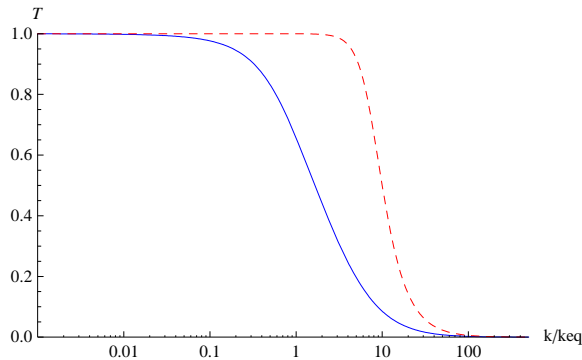
Since the peaks are spherical and isolated, and they are individually assumed to be in the Newtonian regime, their expansion rate is the same as that of a dust FRW universe with the same density, as in the toy model. The fraction of volume which is neither in peaks nor in troughs is taken to expand like the spatially flat matter-dominated FRW model.

The peak number density as a function of time is determined by the power spectrum, which consists of two parts: the primordial power spectrum, determined in the early universe by inflation or some other process, and the transfer function, which describes the evolution between the primordial era and the time when the modes enter the non-linear regime. The transfer function  $T(k)$  simply multiplies the amplitude of the primordial modes. We take a scale-invariant primordial spectrum with the amplitude determined from the CMB anisotropies; small variations from scale-invariance have little effect. For the transfer function, we assume that the dark matter is cold, and we consider two different approximations in order to show the uncertainty in the calculation. The BBKS transfer function [34] is a fit to numerical calculations (we take a baryon fraction of 0.2), and the BDG form introduced in [35] is a simple analytically tractable function with the correct qualitative features. The transfer functions are shown in figure 2, as a function of  $k/k_{\text{eq}}$ , the wavenumber divided by the matter-radiation equality scale. Modes with  $k > k_{\text{eq}}$  enter the horizon during radiation domination, so their amplitude is damped, and the sooner they enter, the more they are damped before the universe becomes matter-dominated, so there is an (approximately)  $k^{-2}$  damping tail. Modes with  $k < k_{\text{eq}}$  enter during the matter-dominated era and retain their original amplitude, and for modes with  $k \sim k_{\text{eq}}$ , the transfer function interpolates between these two regimes. In the BBKS transfer function, the transition is centered around  $k_{\text{eq}}$  and is more smooth, while in the BDG case the transition happens a bit earlier and is more rapid. (Even the more realistic BBKS transfer function has an error of 20–30% compared to Boltzmann codes.)

We have

$$H(t) = \int_{-\infty}^{\infty} d\delta v_{\delta}(t) H_{\delta}(t) , \quad (4.1)$$

where  $d\delta v_{\delta}$  is the fraction of volume in regions with linear density contrast  $\delta$  and expansion rate  $H_{\delta}(t)$ . The correspondence between  $\delta$  and  $H_{\delta}$  is given by the spherical evolution model (i.e. FRW evolution), and the distribution of regions  $v_{\delta}(t)$  is given by the peak statistics, which is determined by the power spectrum of the Gaussian density field. With the transfer function fixed, there are no free parameters: the expansion history  $H(t)$  given by (4.1) is completely determined. Since the primordial spectrum is scale-invariant and the smoothing and peak identification process does



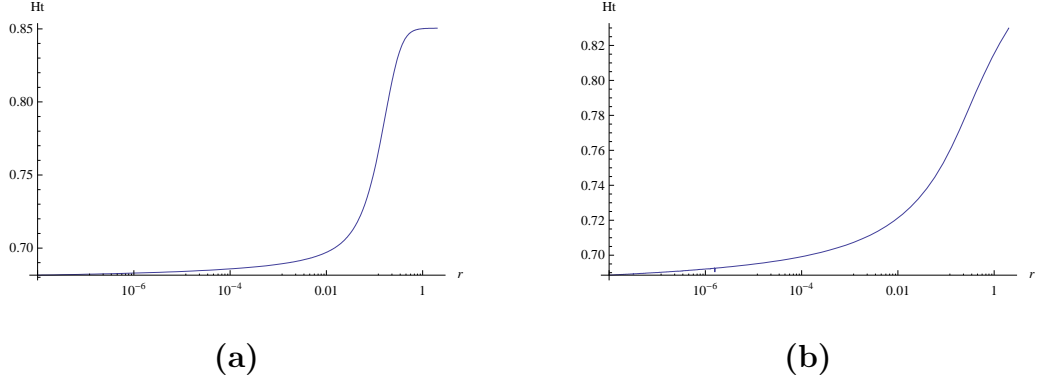
**Figure 2:** The BBKS (blue, solid) and BDG (red, dashed) transfer functions as a function of  $k/k_{\text{eq}}$ .

not introduce a scale, features in the expansion rate as a function of time can only come from the turnover at the matter-radiation equality scale in the transfer function.

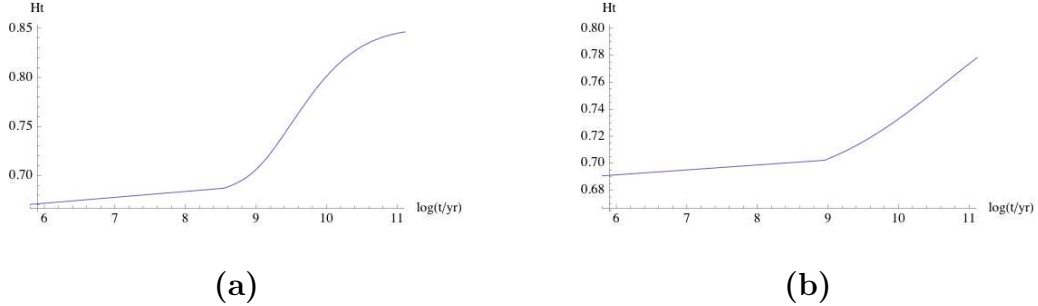
In figure 3 we show  $Ht$  as a function of  $r \equiv k_{\text{eq}}R$ , the smoothing scale relative to the matter-radiation equality scale. Essentially, the coordinate  $r$  is time as measured by the size of the largest generation of structures. We have  $Ht \approx 2/3$  at early times, as the fraction of volume in non-linear structures is small. As time goes on, deeper non-linear structures form, and they take up a larger fraction of the volume. The expansion rate grows (relative to the FRW value) slowly, until there is a rapid rise and saturation, roughly at the scale of matter-radiation equality. It is clear that after  $r = 1$ , when the perturbations which correspond to the matter-radiation equality scale collapse,  $Ht$  must settle to a constant, since the transfer function is nearly unity, and there is no scale in the system anymore.

The matter-radiation equality scale is  $k_{\text{eq}}^{-1} \approx 13.7\omega_{\text{m}}^{-1} \text{ Mpc} \approx 100 \text{ Mpc}$ , using the value  $\omega_{\text{m}} = 0.14$  [1]. Observationally,  $\sigma(t, R) \approx 1$  today on scales somewhat smaller than  $8 h^{-1} \text{ Mpc}$ , so  $R_0 \approx 10 \text{ Mpc}$ . Therefore the present day happens to be located around  $r = 0.1$  in the plots – right in the transition region. Note that nothing related to present day has been used as input in the calculation.

It is instructive to view  $Ht$  also as function of time as measured in years. In figure 4, the horizontal axis is  $\log_{10}(t/\text{yr})$ . For the BDG transfer function,  $Ht$  has the FRW value at one million years, and it grows very slowly until about a billion years, when  $Ht$  starts to rise, and then saturates to a value somewhat larger than 0.8 at some tens of billions of years. For the more realistic BBKS transfer function, the behaviour is qualitatively the same, but the transition is slower and the final value of  $Ht$  is smaller. The slope of the  $Ht$  curve is less steep as a function of time than as a function of  $r$ , because the size of structures grows more slowly at late times. When plotting  $Ht$  as function of the smoothing scale, the comparison scale is  $k_{\text{eq}}$ , whereas here it is the time of matter-radiation equality,  $t_{\text{eq}}$ . Now the amplitude of the primordial perturbations also enters. The timescale follows from the shape of



**Figure 3:** The expansion rate  $Ht$  as a function of  $r = k_{\text{eq}}R$  for (a) the BDG transfer function and (b) the BBKS transfer function.



**Figure 4:** The expansion rate  $Ht$  as a function of time for (a) the BDG transfer function and (b) the BBKS transfer function.

the transfer function. Perturbations which entered the horizon at matter radiation equality reach non-linearity at  $t \approx A^{-3/2}t_{\text{eq}} \approx 100 \text{ Gyr}^5$ , where  $A = 3 \times 10^{-5}$  is the primordial amplitude and the matter-radiation equality time is  $t_{\text{eq}} \approx 1000\omega_{\text{m}}^{-2}$  years  $\approx 50\,000$  years for  $\omega_{\text{m}} = 0.14$ . This is when the expansion rate saturates, and it enters the transition region somewhat earlier.

As noted in section 3, whether or not the expansion accelerates is a quantitative question related to the slope of the  $Ht$  curve. In the present case, while the expansion rate increases relative to the FRW value, the change is not sufficiently rapid for the expansion to accelerate, there is just less deceleration. This is related to the fact that, unlike in the toy model, the overdense regions play almost no role, and the evolution of  $Ht$  can be understood in terms of the underdense voids. At early times, voids take up only a small part of the volume, and  $Ht$  rises smoothly as their volume fraction increases. In order to obtain a more drastic change in  $Ht$ , the expansion rate should have extra deceleration due to overdense regions before the voids take over, as in the toy model. (This effect is present in the model, but it is too small to be visible

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<sup>5</sup>We have  $1 = \sigma(t, R = k_{\text{eq}}^{-1}) = \left( \int_0^{k_{\text{eq}}} \frac{dk}{k} \frac{4}{9} \frac{k^4}{(aH)^4} \Delta_{\Phi}(k) T(k)^2 \right)^{\frac{1}{2}} = \frac{A}{3} \frac{(a_{\text{eq}} H_{\text{eq}})^2}{(aH)^2} = \frac{A}{3} \frac{t^{2/3}}{t_{\text{eq}}^{2/3}}$ , where  $\Delta_{\Phi}(k) = A^2$ , and we have adopted the step function as the window function and approximated  $T(k) = 1$  for  $k \leq k_{\text{eq}}$ .

in figure 3 and figure 4.) If the expansion were to slow down more before the voids take over, the variance and the change in the expansion rate would be larger. The magnitude of the change of  $Ht$  is easy to understand: if the universe were completely dominated by completely empty voids, we would have  $Ht = 1$ . Since not all of the volume is taken up by voids and they are not totally empty,  $Ht$  is somewhat smaller than unity.

It is encouraging that the model gives a change of the right order of magnitude in  $Ht$ , 15–25%, and that the timescale for the change comes out right. However, the model involves many approximations, such as treating structures as spherical, using an approximate transfer function, having an artificial split between the peaks/troughs and the smooth space, not taking into account that the Gaussian symmetry between the overdense and underdense regions is broken in the non-linear regime (in the present treatment, they have equal mass in at all times) and treating the structures as isolated even for small density contrasts and high peak number densities. It is clear the model cannot be trusted beyond an order of magnitude. It is also possible that a more careful statistical treatment would reveal some cancellations that would significantly reduce the effect of backreaction from this approximate estimate.

## 4.2 Beyond Newton

A detailed statistical treatment would require significantly more work than put into the simple semi-realistic model. Before investing the effort, one may ask why bother, since N-body simulations provide detailed information about the non-linear regime of structure formation. The problem is that the simulations use Newtonian gravity with periodic boundary conditions. In Newtonian gravity, the variance and the shear cancel in the backreaction variable  $\mathcal{Q}$  given in (2.10), up to total derivatives which can be written as boundary terms [36]. Boundary terms vanish for periodic boundary conditions, but using a large simulation and considering boxes of the size of the observable universe would not help the situation. Total derivative terms represent a flux, and due to statistical homogeneity and isotropy, the integrated flux over the whole boundary should vanish (up to statistical fluctuations), as otherwise there would be a preferred direction.

In general relativity, the backreaction variable  $\mathcal{Q}$  does not reduce to a boundary term, and the average expansion rate of a volume depends on the behaviour everywhere in the volume, not just on the boundary. In contrast, the Newtonian evolution is sensitive to boundary conditions, even for infinitely far away boundaries, which is related to the fact that Newtonian cosmology does not have a well-posed initial value problem. This is one aspect of the qualitative difference between general relativity and what is called Newtonian cosmology. The small-velocity, weak field limit of general relativity is not Newtonian gravity, as demonstrated by the existence of Newtonian solutions which are not the limit of any general relativity solution [37,38]. Rather, it is a theory with new degrees of freedom and additional

constraints compared to Newtonian gravity [37, 39–42]. The formulation of this limit of general relativity in the cosmological setting with non-linear perturbations is an open issue.

It is important to make sure that an improved statistical treatment would be consistent with the relativistic equations of motion and constraints. One may ask why the average expansion rate in the peak model differs from the FRW result. After all, the individual regions were taken to be Newtonian, and there is at first sight nothing non-Newtonian about the identification of the peaks as structures. In Newtonian gravity, the feature that inhomogeneities do not change the average expansion rate in a statistically homogeneous and isotropic universe can be understood from energy conservation. In the exactly homogeneous and isotropic case, the Newtonian Friedmann equation (multiplied by  $a^2$ ) can be interpreted as stating that the kinetic energy plus the potential energy is constant. The relativistic Friedmann equation is mathematically identical, but has a different physical interpretation, with the constant energy replaced by the spatial curvature term. However, the correspondence does not hold beyond the FRW case. In Newtonian gravity, the total energy is conserved even when the system is inhomogeneous and anisotropic, as long as the force is conservative and the system is isolated (i.e. the boundary terms in  $\mathcal{Q}$  vanish). However, in general relativity, there is no conservation law for the average spatial curvature, and  $a^2\langle^{(3)}R\rangle$  is in general not constant. In the peak model, the average spatial curvature evolves from zero to large negative values as the volume becomes dominated by underdense voids. This is possible in general relativity, while in Newtonian gravity the total energy cannot change.

## 5. Light propagation

### 5.1 The redshift

Let us say we were to have a reliable calculation of the average expansion rate, either via an improved statistical model or by including the relevant general relativistic degrees of freedom in a simulation. What would this tell us about observations? The average is taken on a spacelike hypersurface of simultaneity, but we can only observe things inside the past lightcone. Most cosmological observations are made along the lightcone, measuring the redshift and the angular diameter (or luminosity) distance. In a general spacetime, these quantities are not determined solely by expansion, and certainly not by the average expansion rate along spacelike slices. However, in a statistically homogeneous and isotropic universe where the distribution evolves slowly, the average expansion rate does give the leading behaviour of the redshift and the distance [24, 43]. In a general dust spacetime, the redshift is given by

$$1 + z = \exp \left( \int_{\eta}^{\eta_0} d\eta \left[ \frac{1}{3} \theta + \sigma_{\alpha\beta} e^{\alpha} e^{\beta} \right] \right) , \quad (5.1)$$



where  $\eta$  is defined by  $\partial/\partial\eta \equiv (u^\alpha + e^\alpha)\partial_\alpha$ , and  $e^\alpha$  is the spatial direction of the null geodesic. If there are no preferred directions and the change in the distribution is slow compared to the time it takes for a light ray to pass through a homogeneity scale sized region, the integral over  $\sigma_{\alpha\beta}e^\alpha e^\beta$  is suppressed. In the real universe, the homogeneity scale of around 100 Mpc is indeed much smaller than the timescale for the change in the distribution, which is given by the Hubble scale  $H_0^{-1} = 3000h^{-1}\text{Mpc}$ . In the early universe, structure formation was less advanced, so further down the null geodesic the homogeneity scale is even smaller relative to the Hubble scale. The direction  $e^\alpha$  changes slowly for typical light rays [24], whereas the dust shear is correlated with the shape and orientation of structures, and changes on the length scale of those structures. If there are no preferred directions, structures are oriented in all directions equally over large scales, so  $\sigma_{\alpha\beta}$  should contribute via its trace, which is zero. Therefore the integral over  $\sigma_{\alpha\beta}e^\alpha e^\beta$  should vanish, up to statistical fluctuations and corrections from correlations between  $\sigma_{\alpha\beta}$  and  $e^\alpha$  and evolution of the distribution. We can split the local expansion rate as  $\theta = \langle\theta\rangle + \Delta\theta$ , where  $\Delta\theta$  is the local deviation from the average, and similarly argue that the integral of  $\Delta\theta$  is suppressed relative to the contribution of the average expansion rate.

At this point, the choice of hypersurface is important. For the cancellations to occur, the hypersurface of averaging has to be the hypersurface of statistical homogeneity and isotropy. (In addition, the evolution of the distribution from one hypersurface to another has to be slow compared to the homogeneity scale.) This defines the hypersurface of averaging: the primary quantities are the observable redshift and distance, and averages are useful only insofar as they give an approximate description of what is observed [24, 43]. We have taken the averages on the hypersurfaces of constant proper time of observers comoving with the matter. Since the evolution of structures is governed by the proper time, one can argue that this is close to the hypersurface of statistical homogeneity and isotropy [19, 32, 43]. The details are likely to be more complicated, but non-relativistic changes in the four-velocity which defines the hypersurface lead only to small changes in the averages [24].

Given that  $\langle\theta\rangle = 3\dot{a}/a$ , we obtain  $1+z \approx a(t)^{-1}$ , the same relation between expansion and redshift as in the FRW case. Note that this result depends on the fact that the shear and the expansion rate enter into the integral (5.1) along the null geodesic linearly. In the case of the equations of motion (2.3)–(2.5) for the geometry, the shear and the expansion rate enter quadratically, so the variations do not cancel in the average, and instead we have the generally non-zero backreaction variable  $\mathcal{Q}$ .

## 5.2 The distance

For the angular diameter distance, we can apply similar qualitative arguments to obtain the result [43]

$$H\partial_{\bar{z}} \left[ (1 + \bar{z})^2 H\partial_{\bar{z}} \bar{D}_A \right] \approx -4\pi G_N \langle \rho \rangle \bar{D}_A, \quad (5.2)$$

where  $\bar{D}_A$  is the dominant part of the angular diameter distance with the corrections to the mean dropped, and the same for the redshift,  $1 + \bar{z} \equiv a(t)^{-1}$ . From the conservation of mass, (2.9), it follows that  $\langle \rho \rangle \propto (1 + z)^3$ . The distance is therefore determined entirely by the average expansion rate  $H(z)$  and the normalisation of the density today, i.e.  $\Omega_{m0}$ . For a general FRW model,  $\langle \rho \rangle$  in (5.2) would be replaced by  $\rho + p$ . So the equation for the mean angular diameter distance in terms of  $H(z)$  in a statistically homogeneous and isotropic dust universe (with a slowly evolving distribution) is the same as in the FRW  $\Lambda$ CDM model. If backreaction were to produce exactly the same expansion history as the  $\Lambda$ CDM model, the distance-redshift relation would therefore also be identical. This is the case even though the spatial curvature would be large<sup>6</sup>, as the spatial curvature affects the distances differently than in the FRW case.

Note that in a general spacetime, the luminosity distance is related to the angular diameter distance by  $D_L = (1 + z)^2 D_A$  [37], so (from the theoretical point of view) it measures the same thing.

Backreaction is not expected to produce an expansion history identical to the  $\Lambda$ CDM model: if the expansion accelerates strongly, then this is likely to be preceded by extra deceleration. Therefore the distances will also be different. However, the backreaction distance-redshift relation will be biased towards the  $\Lambda$ CDM model, compared to a FRW model with the same expansion history as in the backreaction case. The reason is that in the FRW model, the equation for  $D_A$  is modified not only by the change in  $H(z)$ , but also by the change in  $\rho + p$ . This may help to explain why distance observations prefer the value  $-1$  for the effective equation of state.

It has been pointed out that the relation between  $D_A(z)$  and  $H(z)$  can be used as a general test of FRW models [44]. If we measure the distance and the expansion rate independently, we can check whether they satisfy the FRW relation. If they do not, the observations cannot be explained in terms of any four-dimensional FRW model. This holds independent of the presence of dark energy or modified gravity, because light propagation depends directly on the geometry of spacetime, regardless of the equations of motion which determine it. Similarly, we can test the backreaction conjecture that the change in the expansion rate at small redshift is due to structure formation without having a prediction for how the expansion rate changes, simply by checking whether the measured  $D_A(z)$  and  $H(z)$  satisfy (5.2). The relation (5.2) and the violation of the FRW consistency condition between expansion and distance is a unique prediction of backreaction which distinguishes it from FRW models. However, the derivation of the relation between  $D_A(z)$  and  $H(z)$  should be done more rigorously, and the expected magnitude of the violation is unclear.

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<sup>6</sup>From (2.7) and (2.8) we see that for  $\Omega_{m0} = 0.3$  and  $q_0 = -0.55$ , corresponding to the spatially flat  $\Lambda$ CDM model with  $\Omega_{\Lambda 0} = 0.7$ , we have  $\langle {}^{(3)}R \rangle_0 = -6.3H_0^2$ , or  $\Omega_{R0} \equiv -\langle {}^{(3)}R \rangle_0 / (6H_0^2) = 1.05 > 1$ . The physical reason for the large spatial curvature is that most of the volume is occupied by very underdense regions.

## 6. Summary

Observations of the universe at late times are inconsistent with homogeneous and isotropic FRW models which have ordinary matter and gravity. The problem is usually addressed by adding exotic matter or modifying general relativity. However, non-linear structures also influence the expansion rate: this is an effect which is present in reality but missing in FRW models. The Buchert equations which do include the effect of structures show that it is possible for the average expansion of a clumpy dust universe to accelerate, and there are toy models which demonstrate this. The physical explanation is simple: faster expanding regions increase their fraction of the volume more rapidly, so the average expansion rate increases. In a semi-realistic model, the correct timescale of about  $10^5 t_{\text{eq}} \sim 10$  billion years and the right order of magnitude for the change of the expansion rate emerge from the physics of structure formation without new parameters.

In Newtonian cosmology, backreaction is necessarily small for a statistically homogeneous and isotropic distribution, but this is not the case in general relativity. Therefore, if backreaction has a significant effect on the expansion rate in the real universe, this is due to non-Newtonian aspects of general relativity.

Even if backreaction is important, the relation between the redshift and the average expansion rate is the same as in FRW models, if the distribution of structures is statistically homogeneous and isotropic and evolves slowly. In contrast, the relation between the average expansion rate and the angular diameter distance is different from the FRW case. This relation is a unique backreaction prediction which makes it possible to distinguish the effect of non-linear structures from FRW dark energy or modified gravity models.

The present estimates of the effect of structure formation on the average expansion rate cannot be trusted beyond an order of magnitude, and it is possible that a careful study will reveal cancellations which lead to a negligible effect. The relation between the average expansion rate and light propagation should also be studied more rigorously, and the difference between general relativity and Newtonian gravity in the cosmological non-linear regime remains to be fully understood. There is much work to be done before we can say whether or not the backreaction conjecture that the failure of ordinary homogeneous and isotropic models at late times is due to the breakdown of homogeneity and isotropy is correct. Until this effect has been quantified, we do not know whether new physics is needed to explain the observations, or if they can be understood in terms of a complex realisation of the physics we already know.

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